HELPING CHILDREN LEARN

mathematics

2ND AUSTRALIAN EDITION

REYS | LINDQUIST | LAMBDIN | SMITH
ROGERS | COOKE | EWING | ROBSON | BENNETT

WILEY

Helping children learn mathematics

SECOND EDITION

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PREFACE

Welcome to the second Australian edition of *Helping Children Learn Mathematics*. This text reflects the ever-changing world of learning and teaching in early childhood to primary level mathematics. Teachers of mathematics not only experience change but, more importantly, have an opportunity to lead the way. This text empowers pre-service teachers to successfully navigate the Australian Curriculum and NAPLAN testing environment with practical, engaging strategies to help children learn mathematics in tomorrow's classroom.

Helping Children Learn Mathematics is built around three main themes:

1. helping children make sense of mathematics
2. incorporating practical experiences and research to guide teaching
3. emphasising major ideas discussed in detail in the Australian Curriculum.

Helping Children Learn Mathematics, 2nd Australian edition, is designed to empower you to help children learn mathematical concepts and skills, as well as important problem-solving strategies. In the process, it will challenge your thinking and, we hope, will further stimulate your interest in learning and teaching mathematics.

The textbook consists of two main parts.

The first part (chapters 1–6) provides a basis for understanding the changing mathematics curriculum in Australia and how children learn mathematics. It offers some guidelines for planning and evaluating instruction. It directs attention to problem solving and assessment, both of which have profound implications for mathematics teaching at all levels, the importance of which is reflected by their integration throughout the book.

The second part (chapters 7–18) discusses teaching strategies and techniques, as well as learning activities, related to specific mathematical topics. The emphasis is on using models and materials to develop concepts and understanding, so that mathematics learning is indeed meaningful.

We believe that meaning is most effectively established by helping students discuss mathematics as they move from concrete materials and examples to generalisations and abstractions.

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Audrey Cooke
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Bronwyn Ewing
August, 2016

FEATURES OF THIS TEXT

We revised this resource in order to meet the changing needs of students preparing to become teachers, or teachers who are continuing to learn. This edition continues the rich tradition of this book with contemporary ideas interspersed. It has been updated to reflect current recommendations from the Australian Curriculum as well as recent research findings relevant to teaching mathematics, while maintaining the characteristics and features that made the US version of the text such a popular choice for education academics. A useable text for instructors, it is also readable and understandable by students who are being introduced to teaching elementary mathematics. Yet, its depth also makes it appropriate for teachers to use as they continue to learn about teaching mathematics.

Snapshot of a lesson features are based on local videos illustrating best practice in mathematics education. The chapter-opener snapshots are excerpts from actual mathematics lessons in Australian classrooms to show the vital role teachers play in helping students talk and learn about mathematics. Featuring student–teacher and student–student dialogue, they cover a wide range of age groups and mathematical topics, demonstrate many effective classroom practices, and provide a smooth, practical segue into the body of the chapter. In addition to strategies woven into the text, the following two sections will help you practise translating theory into practice as you guide and teach children in their study of mathematics.

A wealth of real-world examples from the Australian Curriculum provide a diverse coverage of the most up-to-date issues in the contemporary Australian mathematics classroom.

Concept maps provide clear visual overviews of the content explored in each chapter.

In the classroom activities provide a wealth of ideas and strategies for teaching children the topics of each chapter.

Spotlight on early childhood focuses on the specific teaching and learning needs of early childhood students, providing a comprehensive application of mathematical concepts in real-world classroom scenarios for this critical stage of learning.

Making connections provides a broad scope of connections for pre-service teachers, including a platform for students to explore important cross-cultural issues in the mathematics classroom, as well as coverage of essential material relating to the Australian Curriculum such as proficiency standards, diversity and general capabilities.

A glance at where we've been provides a concise summary of suggested responses to the focus questions posed at the beginning of each chapter.

Things to do embody the text's active learning and teaching approach to mathematics. Divided into two parts, they are designed to engage you in inquiring and thinking about mathematics — to offer you experiences and introduce you to investigations that will help you achieve the understanding and insight you need to be a successful teacher.

Children's literature connections include relevant texts referenced at various points within the text. An annotated list of useful books and websites is provided at the end of each chapter.

ABOUT THIS TEXT

Helping Children Learn Mathematics, 2nd Australian edition, is an ideas text. We believe that you will learn much from reading it and from talking about what you have read. The Australian Curriculum highlights communication as an important part of mathematics learning, and this text is designed to encourage and facilitate communication.

It is not possible — or desirable — to specify the exact steps to follow in teaching mathematics. Too much depends on what is being taught, to whom, and at what levels. In your classroom, it is you who will ultimately decide what to teach, to whom to teach it, how to teach it, and the amount of time to spend. This text will not answer all of these questions for you, but we think you will find it very helpful in making wise decisions as you guide your students in their learning of mathematics. We believe this text will be a valuable teaching resource that you can use again and again in your classroom long after your course has been completed.

CHAPTER 1

School mathematics in a changing world

LEARNING OUTCOMES

1.1 Connecting your disposition towards mathematics with your mathematical experiences.

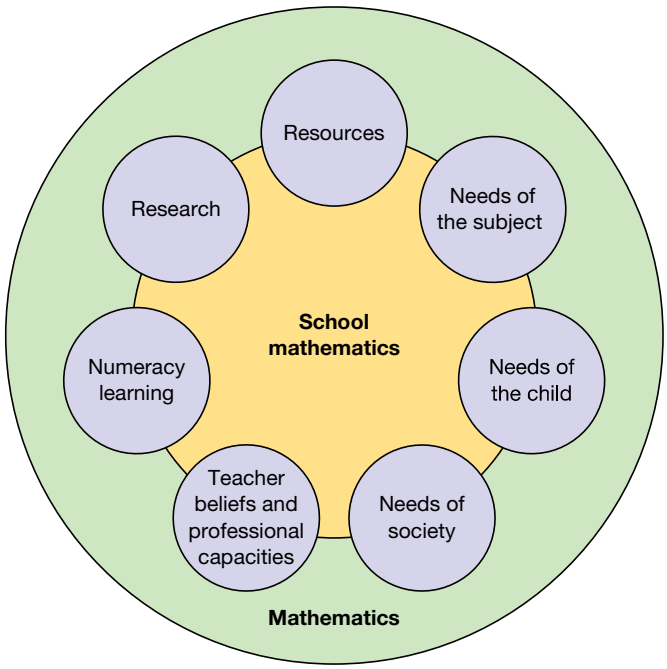
1.2 Recognising factors that determine the mathematics that is currently taught, and the role, rationale and aims of the Australian Curriculum in contemporary mathematics teaching and learning.

1.3 Evaluating resources to determine those of use in your continuing development of mathematics, mathematical learning and teaching mathematics.

'Education is the most powerful weapon which you can use to change the world.'

Nelson Mandela

Chapter 1 concept map



Introduction

Mathematics remains an essential tool for understanding the world, so mathematics education needs to keep pace with the advancements that our society makes at an ever-increasing rate. Taken from a mathematics learning and teaching perspective, the words of Nelson Mandela in the opening quote are very relevant. As a teacher of mathematics, you have the opportunity to provide a positive and meaningful experience for those you help to learn mathematics.

What is your vision of the mathematics you will be teaching? What is your vision of the classroom? Many of you will remember your experiences in primary school. Some of you will remember memorising multiplication tables, operating with fractions or doing long division. Others may remember exploring patterns, doing geometry projects or solving problems. Mathematics in primary school may have been a positive experience for some of you, but for others it was filled with anxiety and frustration. Why were you learning mathematics and when would you use it? Teachers want students to learn mathematics and to recognise that it is a useful subject. How can they make sure this happens?

This text is designed to expand your vision of mathematics teaching and learning and to help you help students learn mathematics. The text interweaves 3 main themes.

- *Theme 1 — recommendations from national professional organisations and from research on learning and teaching mathematics.* These recommendations and research provide a basis for you to understand what mathematics children are expected to learn and how children learn mathematics.
- *Theme 2 — sense making*. Mathematics must make sense to children. If children make sense of the mathematics they are learning, they can build on this understanding to learn more mathematics and use the mathematics to solve problems.
- *Theme 3 — practical suggestions*. Learning to teach mathematics requires experience. This theme is explicated by including many suggestions from teachers and our own experiences as well as many ideas for you to use as you learn to teach and later when you become a teacher.

Learning to teach is a lifelong journey. During that journey, you will often ask questions such as those below.

- What mathematical knowledge and understandings does each student bring to the class?
- What mathematics do students need to learn?
- How can I teach each unique child so that he or she will learn?
- What is the influence of my own attitude toward mathematics?

Your answers to these questions will influence what you do when you are teaching. No matter what the age of the children you teach, you will probably have several general *goals* — for example:

- to help children make sense of specific mathematical content, including both procedures and concepts
- to help children learn how to apply mathematical ideas to solve problems
- to foster positive dispositions, such as persistence, flexibility, willingness to learn and an appreciation of the value of mathematics.

Developing ways to help you reach these 7 goals is considered in later chapters of this text. This first chapter focuses on what mathematics is and what determines the mathematics that is taught in schools. We also share suggestions as to where you can turn for additional help.

1.1 What is mathematics?

LEARNING OUTCOME 1.1 Connecting your disposition towards mathematics with your mathematical experiences.

Frequently, people equate mathematics with arithmetic. In school mathematics, arithmetic is about numbers. At times, arithmetic, especially computation with numbers, has been the only focus in primary school. Currently, however, this limited view of mathematics is not prevalent. Table 1.1 provides an outline of the Australian Curriculum, Assessment and Reporting Authority's (ACARA) *The Australian Curriculum: Mathematics* content strands and sub-strands.

TABLE 1.1 Australian Curriculum content strands and sub-strands for mathematics

Number and Algebra	Measurement and Geometry	Statistics and Probability
Number and place value (F–8)	Using units of measurement (F–10)	Chance (1–10)
Fractions and decimals (1–6)	Shape (F–7)	Data representation and interpretation (F–10)
Real numbers (7–10)	Geometric reasoning (3–10)	
Money and financial mathematics (1–10)	Location and transformation (F–7)	
Patterns and algebra (F–10)	Pythagoras and trigonometry (9–10)	
Linear and nonlinear relationships (7–10)		

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

The 3 strands — Number and Algebra, Measurement and Geometry, and Statistics and Probability — are each composed of sub-strands. Most of these sub-strands are emphasised in the early school years. Within the Number and Algebra strand, the focus progresses from simple to more abstract number representations and computations, and then into algebraic relationships. In the Measurement and Geometry strand, a range of basic concepts and skills are developed across the primary school years, with more

abstract and complex concepts related to reasoning and trigonometry beginning to be attended to in the middle school years. For the Statistics and Probability sub-strands, the emphasis across all of primary school through to the end of lower secondary school reflects an increased awareness of the importance of these areas of mathematics in daily life.

Although we can consider mathematics as a collection of separate areas such as geometry and algebra, this may not be the best way of looking at it. It may be helpful to broaden your view of mathematics. The 5 views below may help you think of mathematics differently than that of being a collection of subjects.

1. *Mathematics is a study of patterns and relationships*. Children need to become aware of recurring ideas and of relationships between mathematical ideas. These relationships and ideas provide a unifying thread throughout the curriculum. Children should come to see how one idea is like or unlike other ideas. For example, children in Year 1 can see how one basic fact (say, $3 + 2 = 5$) is related to another basic fact (say, $5 - 3 = 2$). Older children can relate measuring to the nearest centimetre to rounding to the nearest hundred.
2. *Mathematics is a way of thinking*. Mathematics provides people with strategies for organising, analysing and synthesising information. Often symbolising a real-life problem reduces it to a well-known mathematical procedure, making the problem easier to solve.
3. *Mathematics is an art, characterised by order and internal consistency*. Many children think of mathematics as a confusing set of discrete facts and skills that must be memorised. Teachers tend to focus on developing the skills required to 'do' mathematics, and in doing so they may forget that children need guidance to recognise and appreciate the underlying orderliness and consistency as they construct their own understanding of mathematics.
4. *Mathematics is a language that uses carefully defined terms and symbols*. Learning these terms and symbols enhances our ability to communicate about science, real-life situations and mathematics itself. Like any language, you need to understand the meaning of these words and when it is appropriate to use them.
5. *Mathematics is a tool*. Not only mathematicians use mathematics, but everyone in the course of daily life uses it. Seeing this will help children appreciate why they are learning mathematics. They, too, will be able to use mathematics to solve both abstract and practical problems, just as mathematicians and other people do. Mathematics has become an essential part of our world, both in everyday life and in the workplace.

1.2 What determines the mathematics being taught?

LEARNING OUTCOME 1.2 Recognising factors that determine the mathematics that is currently taught, and the role, rationale and aims of the Australian Curriculum in contemporary mathematics teaching and learning.

Mathematics is highly valued in society by parents, politicians and employers. However, many students do not share this view of mathematics due to a lack of interest and engagement in the subject. Often this is because students are unable to see the relevance or purpose of the subject in a real-life or meaningful way. The *National Numeracy Review Report* (Council of Australian Governments 2008, p. xii) suggests that this scenario is widely held and a risk to Australia achieving its human capital goals. Further, due to the high stakes nature of mathematics achievement as a selection and employment criterion, there is a direct relationship between the level of mathematics studied and students' aspirations and career prospects.

In December 2010, ACARA released *The Australian Curriculum: Mathematics* for implementation in all Australian schools by 2013. This marked the first time in Australia's history that there was a national approach to teaching and learning mathematics, from Foundation to Year 10, across all states and

territories. However, in their review of the Australian Curriculum, Donnelly and Wiltshire (2014) stated 'the implementation picture is extremely confusing and provides little assurance that the Australian curriculum is being implemented, as intended, across the nation' (p. 106). They proposed several reasons for this, including disagreement between ACARA and state and territory governments, with 'a strident argument from states and territories that implementation is their domain and ACARA is not an accountable body for implementation' (p. 107). Donnelly and Wiltshire did find that the Australian Curriculum documents were being incorporated or supplemented by resources developed by the government body responsible for education in that state or territory.

In England, the national curriculum must be delivered by all schools that are maintained by the local authority (Department of Education, 2014). Mathematics, together with English and Science, is considered a core subject (Department of Education, 2014). Teachers are required to 'develop pupils' numeracy and mathematical reasoning in all subjects so that they understand and appreciate the importance of mathematics' (para. 5.2).

In the United States, the National Council of Teachers of Mathematics (NCTM), the largest professional organisation of teachers of mathematics, developed standards for curriculum and for evaluation, teaching and assessment (NCTM 1989, 1991, 1995). Because states and localities in the United States have the right to determine their own school policies, these standards are not prescriptive, but they have provided vision and direction for schools in developing mathematics curricula. In Australia, the national professional body, the Australian Association of Mathematics Teachers (AAMT), has developed several statements and policies pertaining to mathematics learning and teaching. These include *Mathematical Knowledge and Understanding for Effective Participation in Australian Society* (1996a)*, Statement on the Use of Calculators and Computers for Mathematics in Australian Schools* (1996b), *Policy on Numeracy Education in Schools* (1998), *Quality Mathematics in the Middle Years Communiqué* (2006a), *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006b), *Position Paper on Early Childhood Mathematics* (2006), *Position Paper on the Practice of Assessing Mathematics Learning* (2008) and the *Position Paper on Consumer and Financial Literacy in Schools* (2009).

In 2000, the NCTM published an update of the standards in a document titled *Principles and Standards for School Mathematics* (NCTM 2000). The principles represent fundamental beliefs about the characteristics of a high-quality, equitable mathematics program. The standards describe the mathematical content and mathematical processes that should be taught in school mathematics. Combined, the principles and standards present a vision for mathematics education programs in a changing world.

In 2006, the NCTM published *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* as one possible response to the question of how to organise curriculum standards within a coherent, focused curriculum by showing how to 'build on important mathematical content and connections identified for each grade level' (NCTM 2006, p. 3). The tables of focal points and connections are reprinted in appendix B. The rationale, aims and content framework of *The Australian Curriculum: Mathematics* (ACARA 2012) align with those of the NCTM *Principles and Standards for School Mathematics* (2000). Similarly, the guidelines in the *Position Paper on the Practice of Assessing Mathematics Learning* (AAMT 2008) are consistent with those of the NCTM.

It is beneficial to consider the curriculum and standards documents developed in Australia, England, and the US as this will enable you to develop a greater understanding of the content being addressed. It will also enable you to access resources created for each of these 3 locations and to use them in your mathematics programs. As you think about mathematics programs, you should give careful attention to 3 general factors: the needs of the subject, the needs of the child and the needs of society. Consideration of these factors highlights the importance of the guiding principles or the rationale of both the Australian (see figure 1.1) and North American systems. They have many similarities in approach and design.

Content strands
- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

Proficiency strands
- Understanding
- Fluency
- Problem Solving
- Reasoning

Achievement standards
- Indicate the quality of learning that students should typically demonstrate by a particular point in their schooling

Diversity of learners
- Special education needs
- English as an additional language or dialect
- Gifted and talented

General capabilities
- Literacy
- Numeracy
- Information and communication technology (ICT) capability
- Critical and creative thinking
- Ethical understanding
- Personal and social capability
- Intercultural understanding

Cross-curriculum priorities
- Aboriginal and Torres Strait Islander histories and cultures
- Asia and Australia's engagement with Asia
- Sustainability

FIGURE 1.1 Features of *The Australian Curriculum: Mathematics*

Source: © Australian Curriculum, Assessment and Reporting Authority 2015b.

Needs of the subject

The nature of mathematics helps determine what is taught and when it is taught in primary school years. For example, whole numbers are the basis for many mathematical ideas; moreover, experiences with whole numbers arise long before children come to school. Building on children's experiences with counting, the earlier years emphasise whole numbers. Work with fractions and decimals logically follows the work with whole numbers. Such seemingly natural sequences are the result of long years of curricular evolution. This process has involved much analysis of what constitutes a progression from easy to difficult, based in part on what is deemed necessary at one level for the development of ideas at later levels. Once a curriculum is in place for a long time, however, people tend to consider it the only proper sequence. Thus, omitting a topic or changing the sequence of topics often involves a struggle for acceptance. However, research shows that all students do not always learn in the sequence that has been ingrained in our curriculum. You need to be open to change so that each child's needs are met.

Mathematics is continually in flux; new mathematics is created, and new uses of mathematics are discovered. As part of this change, technology has made some mathematics obsolete and has opened the door for other mathematics. The influence of technology and the importance of a flexible approach to curriculum are emphasised in *The Australian Curriculum: Mathematics* (ACARA 2012, p. 11) as part of a student's information and communication technology (ICT) capability:

> Students develop ICT capability as they learn to use ICT effectively and appropriately to access, create and communicate information and ideas, solve problems and work collaboratively in all learning areas at school, and in their lives beyond school. ICT capability involves students in learning to make the most of the technologies available to them, adapting to new ways of doing things as technologies evolve.

You will teach at a time when technology dominates activities both in and out of school. Technology will continue to be important in teaching and learning mathematics, as long as it enhances what is being learned and how it is being taught. As you teach your classes, you should keep asking 3 questions.

1. How can I help children use technology appropriately?
2. What mathematics do children need in order to use technology wisely?
3. What mathematics is no longer necessary because of technology?

Some parents continue to be concerned about the use of calculators in learning mathematics in primary schools. A meta-analysis of 54 research studies on the use of and attitudes toward calculators (Ellington 2003) suggests that using calculators does not hinder the development of mathematical skills and that students who used calculators had better attitudes toward mathematics than those who did not. Of course, children need to learn to use calculators appropriately, as they do any other tool.

No one knows exactly what mathematics will be needed as the twenty-first century progresses, but it is clear that students will need to know how to reason mathematically and how to apply mathematical thinking to a wide range of situations. How you view mathematics will determine how you view teaching mathematics. If you view mathematics as a collection of facts to learn and procedures to practise, then you will teach that to your students. If you view mathematics as a logical body of knowledge, you will adopt teaching strategies that let you focus on guiding children to make sense of mathematics.

Children need a curriculum that does more than represent mathematics as a collection of isolated skills and fun activities. The Australian Curriculum is designed so that 'links between various components of mathematics, as well as the relationship between mathematics and other disciplines, are made clear' (ACARA 2012, p. 3).

Throughout the remaining chapters of this text, you will investigate topics that you can incorporate into the mathematics curriculum for your students.

Needs of the child

The mathematics curriculum has been influenced by beliefs about how children learn and, ultimately, about how they should be taught. Before the early years of the twentieth century, mathematics was taught to train 'mental faculties' or provide 'mental discipline'. Struggling with mathematical procedures was thought to exercise the mind (like muscles are exercised), helping children's brains work more effectively. Around the turn of the twentieth century, 'mental discipline' was replaced by *connectionism*, the belief that learning established bonds, or connections, between a stimulus and responses. This led teachers to the endless use of drills aimed at establishing important mathematical connections.

In the 1920s, the Progressive movement advocated *incidental learning*, reflecting the belief that children would learn as much arithmetic as they needed and would learn it better if it was not systematically taught. The teacher's role was to take advantage of situations as they occurred to teach arithmetic, as well as to create situations in which arithmetic would arise.

During the late 1920s, the Committee of Seven, a committee of school superintendents and principals from midwestern US cities, surveyed pupils to find out when they mastered various topics (Washburne 1931). Based on that survey, the committee recommended teaching mathematics topics according to students' mental age. For example, subtraction facts under 10 were to be taught to children with a mental age of 6 years 7 months, and facts over 10 at 7 years 8 months; subtraction with borrowing or carrying was to be taught at 8 years 9 months. The recommendations of the Committee of Seven had a strong impact on the sequencing of the curriculum in the United States and elsewhere for years afterward.

Another change in thinking occurred in the mid-1930s, under the influence of *field theory*, or *Gestalt theory*. With William A. Brownell (2006) as a prominent spokesperson, this approach placed greater emphasis on a planned program to encourage the development of insight and the understanding of relationships, structures, patterns, interpretations and principles. It contributed to an increased focus on learning as a process that led to *meaning and understanding*. The value of drill was acknowledged, but it was given less importance than understanding; drill was no longer the major means of providing instruction.

The relative importance of drill and understanding is still debated today. In this debate, people often treat understanding and learning skills as if they were opposites, but this is not the case. Clearly, drill is necessary to build speed and accuracy and to make skills automatic. However, equally clearly, you need to know *why* as well as *how*. Both skills and understanding must be developed, and they can be developed together.

Changes in the field of psychology have continued to affect education. During the second half of the twentieth century, educators came to understand that the developmental level of the child is a major factor in determining the sequence of the curriculum. Topics cannot be taught until children are developmentally ready to learn them. Or, from another point of view, topics must be taught in such a way that children at a given developmental level are ready to learn them.

Additionally, educators' attention is being drawn to the increasing evidence that children *construct* their own knowledge. Thus, helping children learn mathematics means being aware of how children construct mathematics from their experiences both in and out of school. Such considerations are now routinely taken into account in developing mathematics curricula.

The Australian Curriculum proficiency strands 'describe the actions in which students can engage when learning and using the content' of mathematics and 'indicate the breadth of mathematical actions that teachers can emphasise' (ACARA 2012, p. 4). Table 1.2 outlines the key student actions that comprise the foci of these strands. These strands should be viewed as interdependent and hence they need to be taught in an interwoven manner.

TABLE 1.2 **Student actions as they relate to proficiency strands**

Proficiency strand	Key ideas
Understanding	• Developing a robust knowledge of adaptable and transferable mathematical concepts • Making connections between related concepts and applying the familiar to develop new ideas.
Fluency	• Developing skills in choosing appropriate procedures • Carrying out procedures flexibly, accurately, efficiently and appropriately • Recalling factual knowledge and concepts readily
Problem Solving	• Making choices • Interpreting, formulating, modelling and investigating problem situations • Communicating solutions effectively
Reasoning	• Developing a capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising

Source: Adapted from © Australian Curriculum, Assessment and Reporting Authority 2015c.

What it means to learn mathematics has changed a great deal over the past century. In a changing world, learning mathematics with understanding is essential in order to meet the goal of mathematical proficiency. Students must actively build knowledge of mathematics from their personal experiences and prior knowledge. Research has shown that if children are able to make sense of the mathematics they are learning, they can build on this understanding to learn more mathematics and use that mathematics to solve problems in order to become mathematically proficient.

Ideas about developing mathematical proficiency are considered in more depth in the next chapter, but here you can consider some initial important questions. What does it mean to learn mathematics with understanding? How did you learn mathematics? How can you learn to understand what children know? Where can you turn to find out what mathematics children need to learn and how you can challenge and support them in learning that mathematics?

To teach mathematics effectively, teachers must know more than just mathematics. They need to know their students as learners and they must adjust their pedagogical strategies in response to students' varying experiences. Teachers must design lessons that reveal to them what students already know, that reveal students' misunderstandings and that guide students to construct more complex understandings of mathematics. Teachers must create challenging and supportive classroom learning environments that help children make sense of mathematics. Teachers must also encourage students to think, question, solve problems and discuss their ideas. 'Effective mathematics teaching requires understanding what students

know and need to learn and then challenging and supporting them to learn it well' (NCTM 2000, p. 16). Teachers can help children make sense of mathematics in many ways. Chapter 3 initiates the discussion of teaching and subsequent chapters focus on ways to teach and on useful types of activity.

Needs of society

The usefulness of mathematics in everyday life and in many vocations has also affected what is taught and when it is taught. In earlier times, mathematics was considered necessary primarily for clerks and bookkeepers. The curriculum was limited to counting; the simpler procedures for addition, subtraction and multiplication; and some facts about measures and fractions. By the late nineteenth century, business and commerce had advanced to the point where mathematics was considered important for everyone. The arithmetic curriculum expanded to include such topics as percentage, ratio and proportion, powers, roots and series.

This emphasis on *social utility*, on teaching what was needed for use in occupations, continued into the twentieth century. The proponents of social utility approaches focused on the essential skills and arithmetic used by carpenters, shopkeepers and other workers as the foundation for teaching school mathematics. They thought that school mathematics programs should be limited to teaching only these skills.

The outburst of public concern in the 1950s over the 'space race' resulted in a wave of research and development in mathematics curricula. Much of this effort was focused on teaching the mathematically talented student. By the mid-1960s, however, concern was also being expressed for the disadvantaged student, as society renewed its commitment to equality of opportunity. With each of these changes, more and better mathematical achievement was promised.

In the 1970s, when it became apparent that the promise of greater achievement had not fully materialised, another swing occurred in curriculum development. Emphasis was again placed on the skills needed for success in the real world. The minimal competency movement stressed the basics. As embodied in sets of objectives and in tests, the basics were considered to be primarily addition, subtraction, multiplication and division with whole numbers and fractions. Thus, the skills needed in colonial times were again being considered by many to be the sole necessities, even though children were now living in a world with calculators, computers and other features of a much more technological society.

By the 1980s, it was acknowledged that no one knew exactly what skills were needed for the future but that everyone needed to be able to solve problems. The emphasis on problem solving matured through the last 20 years of the century to the point where problem solving was not seen as a separate topic but as a way to learn and to use mathematics. In the 1990s, the standards movement became a major focus. Along with standards came an emphasis on assessment, accountability and equity.

Historically, in Australia each state and territory has taken responsibility for developing a K–12 mathematics curriculum program. More recently, guided by the *Melbourne Declaration on Educational Goals for Young Australians* (Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA] 2008), Australia has moved to implementation of a national curriculum from Foundation to Year 12 in specified learning areas.

Mathematics is also considered essential for a competitive society. As the acronym shows, mathematics is an essential element of Science, Technology, Engineering, and Mathematics (STEM) and, as such, is inherent in the strong focus on the importance of STEM education and careers (Office of the Chief Scientist, 2014a). The report on Australia's STEM future by the Office of the Chief Scientist (2014b) recommended that opportunities be provided to enable the development of lifelong skills such as quantitative skills and critical thinking skills (p. 23). In their report on the skills required to ensure Australia is positioned to be both productive and competitive, the Australian Industry Group [AIG] (2013) reiterate the importance of increasing the participation of the population in STEM-related activities.

Assessment

People often think of assessment as testing to see what students have learned. However, it is much broader in scope. Helping all students learn mathematics requires that assessment be an integral part of the teaching program. Assessment should not be something that is done *to* students; rather, a mathematics program must